One-Dimensional Equations for MHD Channel Flows

J. Raeder,* G. Zankl,* and R. Bünde*
Max-Planck-Institut fuer Plasmaphysik, Garching, West Germany

Theme

An analysis is made to clarify the physical meanings of the mean variables, the axial variations of which are described by one-dimensional channel flow equations. The analysis is based on the general conservation equations for mass, momentum, and total enthalpy and leads to a) a list of assumptions necessary to get the commonly used one-dimensional equations, and b) the relations between the distributions of the flow parameters over the channel cross section and the mean values.

Contents

The MHD flow along a channel with rectangular cross section $A(x)$ is considered. All relevant information about the coordinate system, channel geometry, and the directions of the vectors involved is shown in Fig. 1, which presents a channel volume element of length Δx.

A typical set of one-dimensional MHD equations for the treatment of channel flows is given by

$$\bar{\rho}\bar{v}_x A = \dot{m} \quad \text{(equation of continuity)} \tag{1}$$

$$\bar{\rho}\bar{v}_x \frac{d\bar{v}_x}{dx} = -\frac{d\bar{p}}{dx} - \frac{1}{a}(\overline{\tau_{w,a}} + \overline{\tau_{w,c}}) - \frac{2}{b}\overline{\tau_{w,i}} + \overline{(\mathbf{j} \times \mathbf{B})}_x \quad \text{(momentum equation)} \tag{2}$$

$$\bar{\rho}\bar{v}_x \frac{d}{dx}\left(\bar{h} + \frac{\bar{v}_x^{\,2}}{2}\right) = \frac{1}{a}(\overline{q_{w,a}} + \overline{q_{w,c}}) + \frac{2}{b}\overline{q_{w,i}} + \overline{\mathbf{j}\cdot\mathbf{E}} \quad \text{(enthalpy equation)} \tag{3}$$

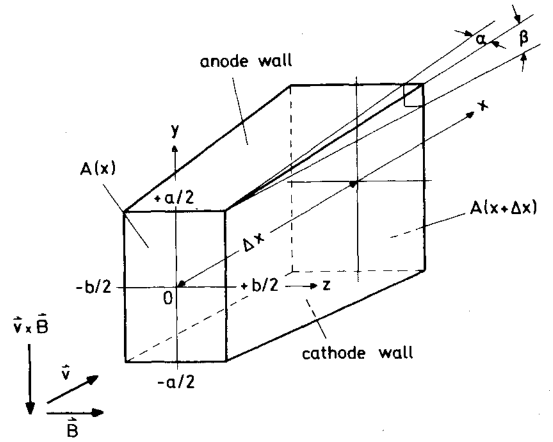

Fig. 1 Volume element of the channel used in the derivation of the one-dimensional equations.

$$\bar{h} = h(\bar{\rho}, \bar{p}) \quad \text{(equation of state)} \tag{4}$$

τ_w = wall shear stress, q_w = density of heat flux to the walls; the indices a, c, and i refer to the anode, cathode, and insulator walls, respectively; the rest of the variables have the usual meanings. The bars indicate that the variables are mean ones. To clarify how these mean variables have to be properly defined, we made an analysis based on the following general magnetogasdynamic equations:

$$\operatorname{div}(\rho\mathbf{v}) = 0 \tag{5}$$

$$\operatorname{div}(\rho\mathbf{v}\mathbf{v} + p\mathbf{1} + \tau) = \mathbf{j} \times \mathbf{B} \tag{6}$$

$$\operatorname{div}\left[\rho\mathbf{v}\left(h + \frac{\mathbf{v}^2}{2}\right) + \mathbf{q} + \tau\cdot\mathbf{v}\right] = \mathbf{j}\cdot\mathbf{E} \tag{7}$$

$$h = h(\rho, p) \tag{8}$$

By integrating Eqs. (5–7) over the volume element shown in Fig. 1 and taking the limit $\Delta x \to 0$, we found the differential equations which describe the axial variations of the fluxes of mass, x-momentum, and total enthalpy through the channel cross section $A(x)$. From these equations, one-dimensional flow equations can be derived, but there are a number of relations which have to be valid for the one-dimensional Eqs. (1–4) to exist at all:

$$\mathbf{v}^2 = v_x^{\,2} \tag{9}$$

$$\tau_{xy} v_y + \tau_{xz} v_z = 0 \tag{10}$$

$$\tau_{xx} = 0 \tag{11}$$

$$q_x = 0 \tag{12}$$

$$\frac{1}{A}\int_C \tan\gamma(s)\cdot[p(s) - \bar{p}]\, ds = 0 \tag{13}$$

C in Eq. (13) indicates contour integration around the cross section A; γ = angle of inclination between a surface line of the channel and the x-axis ($\gamma = \alpha$ on the insulator and $\gamma = \beta$ on the electrode walls); s = length coordinate along the contour.

Equations (9–12) are approximations simplifying the real physical situation: 1) neglect of kinetic energy and friction losses of flows normal to the x-axis [Eqs. (9), (10)]; 2) neglect of the bulk viscosity [Eq. (11)]; and 3) neglect of the axial heat conduction [Eq. (12)].

Relation (13) will be commented on later. If the conditions of Eqs. (9–13) are fulfilled, the following definitions of mean values lead to the one-dimensional Eqs. (1–4) of Ref. 1:

$$\bar{\rho}\bar{v}_x A = \int_A \rho v_x \, dA \tag{14}$$

$$\bar{\rho}\bar{v}_x \bar{v}_x A + \bar{p}A = \int_A (\rho v_x v_x + p)\, dA \tag{15}$$

$$\bar{\rho}\bar{v}_x\left(\bar{h} + \frac{\bar{v}_x^{\,2}}{2}\right)A = \int_A \rho v_x\left(h + \frac{v_x^{\,2}}{2}\right) dA \tag{16}$$

$$\overline{\tau_{w,a}} = \frac{1}{b}\int_{-b/2}^{b/2} \tau_{xy}\left(y = \frac{a}{2}, z\right) dz \tag{17}$$

$$\overline{\tau_{w,c}} = \frac{1}{b}\int_{-b/2}^{b/2} \tau_{xy}\left(y = -\frac{a}{2}, z\right) dz \tag{18}$$

$$\overline{\tau_{w,i}} = \frac{1}{a}\int_{-a/2}^{a/2} \tau_{xz}\left(y, z = \pm\frac{b}{2}\right) dy \tag{19}$$

Received February 20, 1974; synoptic received October 29, 1974. Full paper available from National Technical Information Service, Springfield, Va., 22151 as N75-10365 at the standard price (available upon request).

Index categories: Nozzle and Channel Flow; Plasma Dynamics and MHD.

* Research Scientists.

$$\overline{q_{w,a}} = \frac{1}{b}\int_{-b/2}^{b/2} q_w\left(y = \frac{a}{2}, z\right)dz \tag{20}$$

$$\overline{q_{w,c}} = \frac{1}{b}\int_{-b/2}^{b/2} q_w\left(y = -\frac{a}{2}, z\right)dz \tag{21}$$

$$\overline{q_{w,i}} = \frac{1}{a}\int_{-a/2}^{a/2} q_w\left(y, z = \pm\frac{b}{2}\right)dy \tag{22}$$

$$\overline{(\mathbf{j}\times\mathbf{B})_x} = \frac{1}{A}\int_A (\mathbf{j}\times\mathbf{B})_x\, dA \tag{23}$$

$$\overline{\mathbf{j}\cdot\mathbf{E}} = \frac{1}{A}\int_A \mathbf{j}\cdot\mathbf{E}\, dA \tag{24}$$

The definitions (17–24) represent averaging along a line or over an area. The physical meanings of the mean values $\bar{p}$, $\bar{\rho}$, $\bar{h}$, $\overline{v_x}$ defined by Eqs. (14–16) together with Eq. (4) become clear from the following considerations. We extend the real channel with cross section $A(x)$ downstream from x_o by an ideal channel with constant cross section $A(x_o)$, with ideally smooth, adiabatic walls and zero magnetic induction. The inhomogeneities of the real flow will decay along this ideal channel. The constant values thus developing are just $\bar{p}$, $\bar{\rho}$, $\bar{h}$, and $\overline{v_x}$ because the defining Eqs. (14–16) are the conservation equations for mass, x-momentum, and total enthalpy. Therefore, $\bar{p}$, $\bar{\rho}$, and $\bar{h}$ obviously obey Eq. (4), where the function h establishes between $\bar{\rho}$ and $\bar{p}$ the same mathematical relation as between the local values ρ and p in the equation of state (8). Equations (4), (14–16) have the following properties: a) They define completely the four variables $\bar{p}$, $\bar{\rho}$, $\bar{h}$, and $\overline{v_x}$. b) They ought not to be supplemented by Eq. (13), which would lead to redundancy. Equation (13) therefore describes a real neglect which avoids the occurrence of the left-hand-side term of Eq. (13) in the momentum balance. c) Replacement of, for instance, Eq. (4) by Eq. (13) would destroy the physical meanings of $\bar{p}$, $\bar{\rho}$, $\bar{h}$, and $\overline{v_x}$ previously explained.

Except for a polytropic equation of state, Eqs. (4), (14–16) cannot be solved explicitly for $\bar{p}$, $\bar{\rho}$, $\bar{h}$, and $\overline{v_x}$.[1] In many practical cases the profiles of p, ρ, h, and v_x are nearly rectangular. The deviations from these idealized profiles can be adequately characterized by deficiency thicknesses such as:

$$\delta_d = \int_0^\delta (1-\rho v_x/\rho_o v_{xo})\, dy \quad \text{(displacement thickness)} \tag{25}$$

$$\delta_m = \int_0^\delta (1-v_x/v_{xo})\rho v_x/\rho_o v_{xo}\, dy \quad \text{(momentum thickness)} \tag{26}$$

$$\delta_h = \int_0^\delta [1-(h-h_w)/(h_o-h_w)]\rho v_x/\rho_o v_{xo}\, dy \quad \text{(enthalpy thickness)} \tag{27}$$

(δ = boundary-layer thickness). The first-order solution of Eqs. (4, 14–16) with respect to δ_d, δ_m, and δ_h is given by:

$$\bar{p} = \overline{p^0} + \Delta p = \frac{1}{A}\int p\, dA + \Delta p \tag{28}$$

$$\bar{\rho} = \overline{\rho^0} + \Delta\rho = \frac{(\int \rho v_x\, dA)^2}{A\int(\rho v_x)v_x\, dA} + \Delta\rho \tag{29}$$

$$\bar{h} - h_w = (\overline{h^0} - h_w) + \Delta h = \frac{\int \rho v_x (h-h_w)\, dA}{\int \rho v_x\, dA} + \Delta h \tag{30}$$

$$\bar{v}_x = \overline{v_x{}^0} + \Delta v_x = \frac{\int(\rho v_x)v_x\, dA}{\int \rho v_x\, dA} + \Delta v_x \tag{31}$$

$$\Delta p = -(\overline{\rho^0 v_x{}^0})^2\,\Delta/N \tag{32}$$

$$\Delta\rho = -\overline{\rho^0}^2\,\Delta/N \tag{33}$$

$$\Delta h = -\overline{\rho^0 v_x{}^0}^2\,\Delta/N \tag{34}$$

$$\Delta v_x = \overline{\rho^0 v_x{}^0}\,\Delta/N \tag{35}$$

$$\Delta = \overline{h^0} - h(\overline{p^0}, \overline{\rho^0}) \tag{36}$$

$$N = \overline{\rho^0 v_x{}^0}^2 - \overline{\rho^0}^2\overline{v_x{}^0}^2(\partial h/\partial p)_\rho - \overline{\rho^0}^2(\partial h/\partial\rho)_p \tag{37}$$

Equations (28–31) show that the mean variables are given by fairly complicated integrals containing ρv_x as a kind of weighting function. Therefore, $\bar{h}$, for instance depends not only on the spatial distribution of h, but also on the profiles of p, ρ, and v_x.

In many practical applications, it is possible to subdivide the flow into a core (with the values p_o, ρ_o, h_o, and v_{xo}) and thin boundary layers. The integrals in Eqs. (28–31) may then be approximated by expressions of the type:

$$\int \rho v_x\, dA \approx \rho_o v_{xo} A\left(1 - \frac{1}{a}\delta_{d,a} - \frac{1}{a}\delta_{d,c}\right)\left(1 - \frac{2}{b}\delta_{d,i}\right) \tag{38}$$

$$\int \rho v_x{}^2\, dA \approx \rho_o v_{xo}{}^2 A\left(1 - \frac{1}{a}\delta_{m,a} - \frac{1}{a}\delta_{m,c} - \frac{1}{a}\delta_{d,a} - \frac{1}{a}\delta_{d,c}\right)\times \left(1 - \frac{2}{b}\delta_{m,i} - \frac{2}{b}\delta_{d,i}\right) \tag{39}$$

$$\int \rho v_x (h-h_w)\, dA \approx \rho_o v_{xo}(h_o-h_w)A \times \left(1 - \frac{1}{a}\delta_{h,a} - \frac{1}{a}\delta_{h,c} - \frac{1}{a}\delta_{d,a} - \frac{1}{a}\delta_{d,c}\right)\left(1 - \frac{2}{b}\delta_{h,i} - \frac{2}{b}\delta_{d,i}\right) \tag{40}$$

By using such formulas, it is possible to derive explicit expressions for $\overline{p^0}$, $\overline{\rho^0}$, $\overline{h^0}$, $\overline{v_x{}^0}$, and Δ if the actual form of the equation of state (8) is given. From these results, the mean values $\bar{p}$, $\bar{\rho}$, $\bar{h}$, and $\overline{v_x}$ can be calculated by using Eqs. (28–37).

The Eqs. (4), (14–16) can be interpreted as rules how to compare measured and calculated values: measured profiles of p, ρ, v_x, and h have to be integrated according to Eqs. (14–16) by using Eq. (8), thus yielding $\bar{p}$, $\bar{\rho}$, $\bar{h}$, and $\overline{v_x}$. Only the comparison of these values with those calculated one-dimensionally is physically meaningful.

The differences between the mean values and, for instance, values in the core flow are of special importance in MHD applications. For example, the value σ_o of the electrical conductivity in the core deviates by a typical factor 2 from that determined by $\bar{h}$.

References

[1] Dzung, L. S., "Konsistente Mittelwerte in der Theorie der Turbomaschinen für kompressible Medien," *Brown Boveri Mitteilungen*, Vol. 10, Oct. 1971, pp. 3–10.